The PJS technique and the construction of the first origami level-4 Menger sponge

Serena Cicalò
CDO, Italy
serena.menger4@gmail.com

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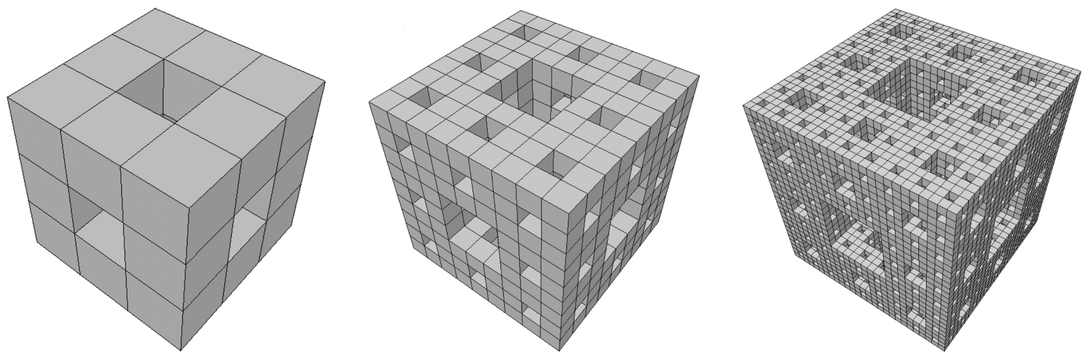
Introduction

The **Menger sponge**, first described by Karl Menger in 1926, is a 3-dimensional fractal with zero volume and infinite area. The *first level* is obtained by dividing a cube into 27 smaller cubes after which the cube in the middle and those at the center of each face are removed (20 cubes remain). To obtain the *second level* we repeat the process with each new cube. Again, for the *third level*, we repeat it for all 20^2 cubes of the second level, and so on.



Repeating this process infinitely eventually gives the Menger sponge fractal.

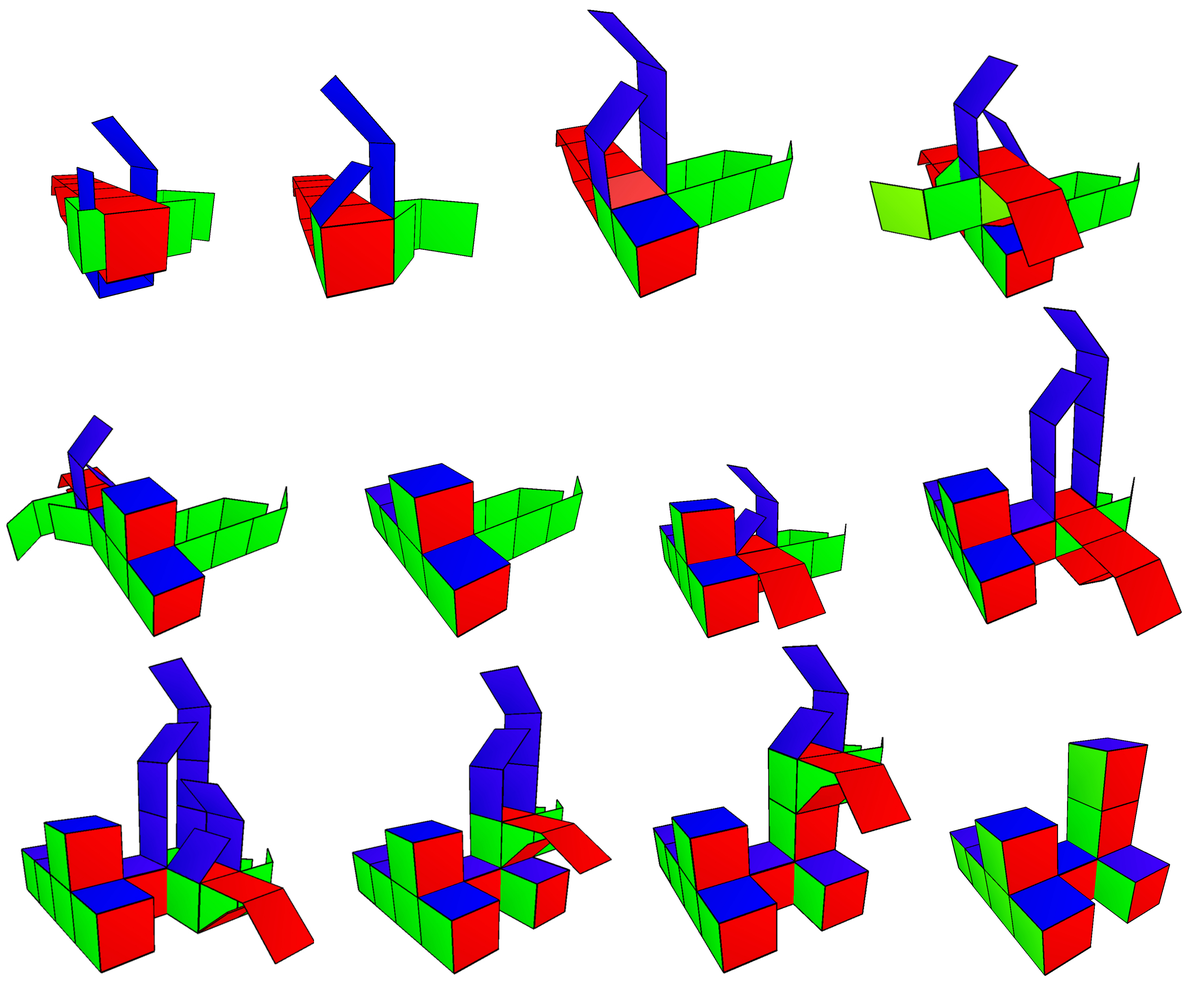
The level-n Menger sponge is a **polycube**, that is a polyhedron composed by elementary cubes, called *units*, connected face-to-face.

Origami Menger sponges

The first level-3 **origami Menger sponge** was completed in 2005 by *Jeannine Mosely* [1] with 200 volunteers. Her method consisted in making single cubes by folding business cards and then joining them together. Several level-3 sponges have been constructed since then, but nobody ever managed to construct a level-4 origami Menger sponge. This would require more than one million business cards and produce a cube weighing more than a tonne, so that the lateral connections probably would not be able to support the weight.

PJS technique

The **PJS technique** (where PJS stands for "*pleat and join strips*") can be used to construct polycubes. With the PJS technique, every unitary cube arises from the intersection of three paper strips in the three spatial directions. Each strip, pleated in squares, has to cover four faces of a tower of stacked units, that I call a *segment*, having as length the number of units that form the tower. If the strip has to create a segment of length k, then it has to be pleated into a *module* of $2k+4$ squares. See the following figure as an example.



Since all the units making a segment are covered by a single strip, the structures built with the PJS technique are much more stable and solid than those made just by face-to-face connections of single cubes.

Menger sponge as a polycube

I found a *formula* to compute the number of segments that make it up in all directions and, consequently, the number of modules needed for each length to build the level-n Menger sponge with the PJS technique.

Theorem. *The level-n Menger sponge is formed, in every direction, of:*

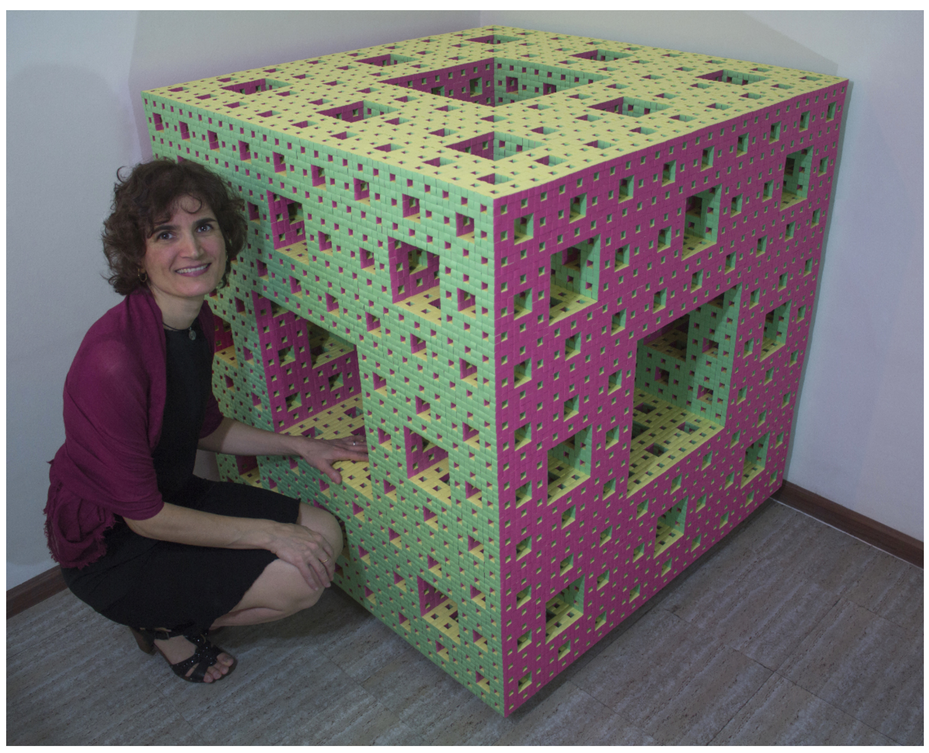
- 4^n *segments of length* 3^n,
- $\frac{2}{3}\cdot(5^{k-1}-2^{k-1})\cdot 4^n$ *segments of length* $2\cdot 3^{n-k}$ *for every* $k=2,\dots,n$,
- $\frac{2}{3}\cdot(5^{k-1}+2^k)\cdot 4^n$ *segments of length* 3^{n-k} *for every* $k=1,\dots,n$.

Example. The number of segments of a level-4 Menger sponge, in each direction, along with their lengths is given by the table:

length	number of segments
81	4^4
27	$2\cdot 4^4$
18	$2\cdot 4^4$
9	$6\cdot 4^4$
6	$14\cdot 4^4$
3	$22\cdot 4^4$
2	$78\cdot 4^4$
1	$94\cdot 4^4$

Level-4 origami Menger sponge

With the PJS technique I built the *first* level-4 origami Menger sponge.



I used around 21 km of paper strips, 1.2 cm wide, in three different colors. The edges are slightly more than 1 m long and it weights approximately 25 kg. It took me 15 months of work. It was completed on November 13th, 2016.

Concluding remarks

- The construction of a level above the fourth is difficult because it would take a very long time. This is due to the fact that the PJS technique, while guaranteeing the stability and solidity of the result, is not suitable for collaborative projects.
- I realized several level-2 and level-3 Menger sponges. My last work has been the construction of a level-3 Menger sponge using strips of width 5 mm. The resulting cube has edges of length a bit less than 15 cm. Note that using smaller strips makes the construction much more difficult.
- It is likely that the PJS technique, that I created to realize polycubes in origami, can also be adapted for the construction of structures made by thin and malleable materials other than paper.

References

[1] Jeannine Mosely. Crowdsourcing origami sculptures. In *Origami6 : II. Technology, Art, Education*, pages 625–634. AMS, 2015.

Acknowledgements

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